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Physical spectrum of conformal SU(N) gauge theories

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We investigate the physical spectrum of vector-like SU(N) gauge theories with infrared coupling close to but above the critical value for a conformal phase transition. We use dispersion relations, the momentum dependence of the dynamical fermion mass and resonance saturation. We argue that the second spectral function sum rule is substantially affected by the continuum contribution, allowing for a reduction of the axial-vector–vector mass splitting with respect to QCD-like theories. Possible consequences for technicolor theories are described. [S0556-2821(99)00606-2]

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The past few years have seen renewed interest in the study of the physical properties of gauge field theories with an infrared fixed point [1]. While ordinary QCD does not fall into this category, other theories of great interest do. The study of $N=1$ supersymmetric gauge theories has lead to a reasonable picture for the different phases depending on the number N_f of matter multiplets [1]. For a range of N_f, an infrared fixed point exists and the theory is in the ''non-Abelian coulomb phase.'' Even ordinary gauge theories can contain infrared fixed points depending on the number of matter fields. In this Brief Report we consider an SU(N) gauge theory with N_f flavors, whose quantum symmetry group is $SU_L(N_f)\otimes SU_R(N_f)\otimes U_V(1)$. It is well known that if N_f is large enough, but below $11N/2$, an infrared fixed point α_* exists, determined by the first two terms in the renormalization group β function. For N_f near $11N/2$, α_* is small and the global quantum symmetry group remains unbroken. For small N_f, on the other hand, we expect the chiral symmetry group $SU_L(N_f)\otimes SU_R(N_f)$ to break to its diagonal subgroup.

It is an important and unsolved problem to determine where the phase transition takes place as N_f is varied. One possibility is that it happens at a relatively large value of $N_f/N(\approx 4)$ corresponding to an infrared fixed point accessible in perturbation theory [2]. An alternative possibility is that the transition takes place in the strong coupling regime, corresponding to relatively small values of N_f/N [3].

The larger value emerges from studies of the renormalization group (RG) improved gap equation, and corresponds to the perturbative infrared fixed point α_* reaching a certain critical value α_c. These studies also show that the order parameter, for example the Nambu-Goldstone boson decay constant F_π, vanishes continuously at the transition. A recent analysis [4] indicates that instanton effects could also trigger chiral symmetry breaking at comparably large values of N_f/N. It has been noted [2] that in such a transition, there are no light degrees of freedom in the symmetric phase other than the fermions and gluons. In the broken phase near the transition, the approximate conformal symmetry suggests that all massive states scale to zero with the order parameter [5]. It follows that a simple Ginzburg-Landau Lagrangian cannot be used to explore the transition.

Here, we will study the spectrum of states in the broken phase near a large N_f/N transition, in particular the possibility that parity near-degeneracy or even inversion takes place. This could have consequences for electroweak symmetry breaking, since near-critical gauge theories provide a natural framework for walking technicolor theories [6].

To set the stage for examining the spectrum of states, we note that in a near-critical theory governed by an infrared fixed point the coupling is near the fixed point for momenta ranging from the small scale associated with the physical states up to some intrinsic, renormalization scale Λ. In Ref. [2], this scale was defined such that $\alpha_\Lambda\equiv\alpha(\Lambda)\approx 0.78\alpha_*$. Above this scale, asymptotic freedom sets in.

A natural framework for exploring the relation between the infrared fixed point behavior and the spectrum of light states is provided by the Weinberg sum rules.[1] The relevant two point Green function is the time honored vector-vector minus axial-vector–axial-vector vacuum polarization, known to be sensitive to chiral symmetry breaking. We define

$$i\Pi^{a,b}_{\mu\nu}(q)\equiv\int d^4x\, e^{-iqx} \times[\langle J^a_{\mu,V}(x)J^b_{\nu,V}(0)\rangle-\langle J^a_{\mu,A}(x)J^b_{\nu,A}(0)\rangle], \tag{1}$$

where

$$\Pi^{a,b}_{\mu\nu}(q)=(q_\mu q_\nu-g_{\mu\nu}q^2)\,\delta^{ab}\Pi(q^2). \tag{2}$$

Here $a,b,=1,\dots,N_f^2-1$, label the flavor currents and the SU(N_f) generators are normalized according to $\mathrm{Tr}[T^aT^b]=(1/2)\delta^{ab}$. The function $\Pi(q^2)$ obeys the unsubtracted dispersion relation

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[1] de Rafael and Knecht [7] have recently used the Weinberg sum rules to study the ordering pattern of states in large-N QCD with N_f fixed. Since there is no infrared fixed point in that limit, their results are not relevant to the problem we are studying.

$$\frac{1}{\pi}\int_0^\infty ds\,\frac{\mathrm{Im}\,\Pi(s)}{s+Q^2}=\Pi(Q^2), \tag{3}$$

where $Q^2=-q^2>0$, as well as the constraint [8] $-Q^2\Pi(Q^2)>0$ for $0<Q^2<\infty$.

Because the theory exhibits asymptotic freedom above Λ, the behavior of $\Pi(Q^2)$ at asymptotically high momenta is the same as in ordinary QCD, i.e., it behaves like Q^{-6} [9]. Expanding the left hand side of the dispersion relation thus leads to the two conventional spectral function sum rules

$$\frac{1}{\pi}\int_0^\infty ds\,\mathrm{Im}\,\Pi(s)=0,\quad \frac{1}{\pi}\int_0^\infty ds\,s\,\mathrm{Im}\,\Pi(s)=0. \tag{4}$$

The approximate conformal symmetry at scales below Λ^2 will mean, however, that the second of these integrals cannot be saturated by a simple set of low lying resonances. A modified second Weinberg sum rule will emerge.

We break the integration into the region of the low lying resonances and the region from there up to Λ^2. (The contribution from beyond Λ^2 will be negligible.) The scale of the lower region is set by the dynamical mass of the fermion $\Sigma(p)$, which has a zero-momentum value $\Sigma(0)$, taken here to be positive, and falls with increasing Euclidean momentum. The dynamical mass is related to F_π by $\Sigma(0)\approx 2\pi F_\pi/\sqrt{N}$. [$\Sigma(p)$ is of course not a gauge invariant quantity [10], but this order of magnitude relation is true in a wide class of gauges.] The first region extends from *zero* to a continuum threshold s_0 which we expect to be on the order of twice the dynamically generated fermion mass: $s_0=O(4\Sigma^2(0))$. In this regime, the integral is saturated by the Nambu-Goldstone pseudoscalar along with massive vector and axial-vector states. If one assumes, for example, that there is only a single, zero-width vector multiplet and a single, zero-width axial vector multiplet, then

$$\mathrm{Im}\,\Pi(s)=\pi F_V^2\delta(s-M_V^2)-\pi F_A^2\delta(s-M_A^2)-\pi F_\pi^2\delta(s). \tag{5}$$

As discussed above, we will take all the masses and widths to scale to 0 with F_π at the transition. All are therefore small with respect to the intrinsic scale Λ near the transition.

The second region, extending from s_0 up to Λ^2, is associated with the continuum, and encodes the conformal properties of the theory. In this ''conformal region'' we estimate the contribution to $\mathrm{Im}\,\Pi(s)$ by evaluating the relevant Feynman diagrams for the vacuum polarization in the presence of a dynamically generated fermion mass, and show that it can substantially affect the low lying mass spectrum through the second sum rule of Eq. (4). We first compute $\Pi(Q^2)$ for Euclidean momentum and then continue analytically.

We approximate the Euclidean computation in this range by a single loop of fermions (quarks) with dynamical mass $\Sigma(p)$, with Ward identities respected but with additional perturbative corrections neglected. The framework is similar to that of ''dynamical perturbation theory,'' sometimes employed to compute the parameters of the low energy chiral Lagrangian [11]. Here, however, the approximation is employed only at the higher momentum scales of the conformal region. Even though this is still sub-asymptotic (below Λ), it is plausible that the approximation is more reliable than at low energies where confinement sets in and the resonances appear.

To summarize, we take the spectrum to consist of a set of low lying resonances with masses on the order of $\Sigma(0)\approx 2\pi F_\pi/\sqrt{N}$, along with a continuum of quarks and gluons, with interaction strength given approximately by the infrared fixed point, extending from there up to Λ^2 (the conformal region).

In the conformal region, $\Sigma(p)$ is determined by the RG-improved gap equation. Instanton effects should be negligible in this range since large instantons (of order the inverse dynamical mass) are most important in chiral symmetry breaking. The running coupling $\alpha(p)$ falls slowly from α_* throughout the conformal region. The form of $\Sigma(p)$ depends on whether p is below or above the scale Λ_c at which $\alpha(p)$ passes through α_c. This scale approaches 0 at the transition as $\Lambda_c/\Lambda\sim(\alpha_*-\alpha_c)^{1/b\alpha_*}$, where b is the coefficient of the first order term in the β function. Below Λ_c, the solution can be written in the approximate form [12]

$$\Sigma(p)=\frac{\Sigma(0)^2}{p}\sin\left(\int_{O(\Sigma(0))}^{p}\frac{dk}{k}\sqrt{\frac{\alpha(k)}{\alpha_c}-1}+\phi\right). \tag{6}$$

We have taken the lower limit of integration to be of order $\Sigma(0)$ where nonlinearities enter and change the form of the solution. We have dropped terms explicitly involving derivatives of $\alpha(k)$ since the coupling is near the fixed point in this regime. Note that the sin function remains positive providing the argument is less than π. The nonvanishing positive phase $\phi\,(<\pi)$ insures that for momenta of order $\Sigma(0)$ the dynamical mass is nonzero and of order $\Sigma(0)$.

The character of the solution changes above Λ_c. It is a positive definite monotonically decreasing function, continuously connected to the lower solution [Eq. (6)] at Λ_c. We have not derived a general closed form in this range, but a qualitatively correct form can be obtained by again neglecting terms explicitly involving derivatives of $\alpha(p)$. $\Sigma(p)$ can then be written in the form

$$\Sigma(p)=A\frac{\Sigma(0)^2}{p}\sinh\left(\delta-\int_{\Lambda_c}^{p}\frac{dk}{k}\sqrt{1-\frac{\alpha(k)}{\alpha_c}}\right), \tag{7}$$

where A and δ are two positive definite constants of order unity.

With δ large enough, $\Sigma(p)$ will be positive even at the upper end of this region ($p=\Lambda$).

Imposing the continuity of $\Sigma(p)$ at $p=\Lambda_c$ and using the fact that $\alpha(k)\to\alpha_*$ for small k leads to the critical behavior $\log(\Lambda_c/\Sigma(0))\sim(\alpha_*/\alpha_c-1)^{-1/2}$, and therefore also to $\log(\Lambda/\Sigma(0))\sim(\alpha_*/\alpha_c-1)^{-1/2}$.

We next use these results to compute $\Pi(Q^2)$ for Euclidean momentum and then derive the form of $\mathrm{Im}\,\Pi(s)$ throughout the conformal region by analytic continuation. To do this, we make a further simplification that does not change the qualitative behavior and allows the integrations to be done analytically. In the region below Λ_c, we take $\alpha(p)$ to be constant and equal to the fixed point value $\alpha_* \; (>\alpha_c)$. The absorptive part $\mathrm{Im}\,\Pi(s)$ then takes the form

$$\mathrm{Im}\,\Pi(s) = N\frac{9\Sigma(0)^4}{16 s^2\pi^2}\sinh(\eta_*\pi)\sin\left(\eta_*\ln\frac{s}{s_0}+2\phi\right) \quad (8)$$

for $s_0 < s < \Lambda_c^2$, where $\eta_* = (\alpha_*/\alpha_c - 1)^{1/2}$.

In the region well above Λ_c, we take $\alpha(p)$ to be constant and equal to $\alpha_\Lambda \equiv \alpha(\Lambda) \approx 0.78\alpha_* \; (<\alpha_c)$. The argument of the sinh [Eq. (7)] then becomes $\delta - \eta_\Lambda \ln(p/\tilde{\Lambda})$, where $\Lambda_c < \tilde{\Lambda} < p < \Lambda$ and $\eta_\Lambda = (1-\alpha_\Lambda/\alpha_c)^{1/2} \simeq 0.47$. The ratio $\tilde{\Lambda}/\Lambda$ is nonzero in the limit $\alpha_\star \to 0$. The absorptive part $\mathrm{Im}\,\Pi(s)$ is then

$$\mathrm{Im}\,\Pi(s) = -NA^2\frac{\Sigma(0)^4}{4 s^2\pi^2}\sin(\eta_\Lambda\pi)[e^{2\delta-\eta_\Lambda\ln(s/\tilde{\Lambda}^2)}\gamma(\eta_\Lambda) - e^{-2\delta+\eta_\Lambda\ln(s/\tilde{\Lambda}^2)}\gamma(-\eta_\Lambda)], \quad (9)$$

where it can be shown that $\gamma(-\eta_\Lambda)/\gamma(\eta_\Lambda) \leq 1$ for any $0 \leq \eta_\Lambda < 1$.

For s approaching Λ_c^2, $\mathrm{Im}(s)$ is negative and vanishingly small. Throughout the conformal region, $\mathrm{Im}\,\Pi(s)$ behaves like $1/s^2$ times a function that oscillates from positive to negative in the range up to Λ_c^2, and then remains negative from Λ_c^2 up to Λ^2. The negativity of $\mathrm{Im}\,\Pi(s)$ above Λ_c^2 [Eq. (9)] is insured by the same condition, $2\delta > \eta_\Lambda \ln(s/\tilde{\Lambda}^2)$, that guarantees the positivity of $\Sigma(p)$ in this region. That $\mathrm{Im}\,\Pi(s)$ [Eq. (8)] can oscillate in the region below Λ_c^2 even though $\Sigma(p)$ [Eq. (6)] does not, is insured by the fact that the argument of the sin in Eq. (8) involves the log of a squared momentum. Note that close to the phase transition ($\eta_* \to 0$), the imaginary part for $s_0 < s < \Lambda_c^2$ is suppressed relative to the imaginary part well above Λ_c by a factor of η_*. This is true even at the high end of the lower range where $\eta_* \ln(s/s_0) = O(1)$. This suppression factor will be compensated by the integration weight in the second spectral function sum rule.

Armed with this information, we now examine the spectral function sum rules. We first note that the first sum rule [Eq. (4)] is concentrated at low momenta. The contribution from the region from s_0 to Λ^2 is suppressed relative to the overall mass scales by the small factor η_* or by large inverse masses. If the low lying resonances are represented as in Eq. (5), one obtains the familiar result

$$F_V^2 - F_A^2 = F_\pi^2. \quad (10)$$

A more general representation of the resonance spectrum would replace the left hand side of this relation with a sum over vector and axial states.

The second sum rule [Eq. (4)] receives important contributions from throughout the conformal region. If a single vector and a single axial vector are used to represent the low lying resonances, we find

$$F_V^2 M_V^2 - F_A^2 M_A^2 = N\,9\frac{\Sigma(0)^4}{16\pi^2}\left[\cos\left(\eta_*\ln\frac{\Lambda_c^2}{s_0}+2\phi\right) - \cos 2\phi + 4A^2\frac{\sin(\eta_\Lambda\pi)}{9\,\eta_\Lambda\pi}F(\eta_\Lambda,\delta)\right], \quad (11)$$

where $F(\eta_\Lambda,\delta)$ can be shown to be positive and $O(1)$.

The quantity in square brackets arises from both above and below Λ_c^2 and its specific form depends on the approximations we have made. For our purposes, however, it is enough to know that it is positive and $O(1)$. This is clearly true for the third term, arising from above Λ_c^2. The sum of the first two terms, arising from below Λ_c^2, is also positive as long as $2\phi + \eta_* \ln(\Lambda_c/\sqrt{s_0}) > \pi$ (a wide range of values). The second sum rule can then be written in the form

$$F_V^2 M_V^2 - F_A^2 M_A^2 = 2a\Sigma(0)^2 F_\pi^2, \quad (12)$$

where a is expected to be positive and $O(1)$, and where we have used the relation $\Sigma(0) \approx 2\pi F_\pi/\sqrt{N}$. As in the case of the first sum rule, a more general resonance spectrum will lead to a left hand side with a sum over vector and axial states. In either case, the conformal region enhances the vector piece relative to the axial.

Combining the two sum rules, and for simplicity restricting to the single vector and axial vector spectrum, leads to

$$M_A^2 - M_V^2 \simeq \frac{F_\pi^2}{F_A^2}[M_V^2 - 2a\Sigma(0)^2]. \quad (13)$$

The conformal region is thus expected to give a negative contribution to the axial-vector mass difference, not present in QCD-like theories. Since $2a\Sigma(0)^2$ is of order M_V^2, the two states will be closer in mass than in QCD-like theories without an extended conformal region. The vector-axial mass pattern might even be inverted with respect to QCD. Note that this is an asymptotic result as the critical value of N_f is approached from below. In this limit, $\Sigma(0)$ and all the resonance masses are of course vanishing relative to the intrinsic scale Λ. Equation (13) says that the splitting relative to this overall scale is further reduced. We would come to a qualitatively similar conclusion if a more general representation of the resonance spectrum were used. This general feature of a near-critical gauge theory governed by an infrared stable fixed point (a walking theory) is our principal result.

Next we note that this changed spectrum of states could have consequences in technicolor theories since the kind of near-critical theory discussed above could provide a natural framework for a walking technicolor theory [6]. The S parameter [13] represents an important test for any technicolor theory. It is related to the absorbitive part of the vector-vector minus axial-vector–axial-vector vacuum polarization as follows [13]:

$$S = 4\int_0^\infty \frac{ds}{s}\,\mathrm{Im}\,\overline{\Pi}(s) = 4\pi\left[\frac{F_V^2}{M_V^2} - \frac{F_A^2}{M_A^2}\right], \quad (14)$$

where $\mathrm{Im}\,\overline{\Pi}$ is obtained from $\mathrm{Im}\,\Pi$ by subtracting the Goldstone boson contribution, and where the final expression comes from using the single vector and single axial vector representation of the resonance region (where the spectral integral is strongly dominated). By using the form of the physical spectrum given by Eq. (13), we get

$$S \simeq 4\pi F_\pi^2 \left[\frac{1}{M_V^2} + \frac{1}{M_A^2} - \frac{2a\Sigma(0)^2}{M_V^2 M_A^2} \right], \qquad (15)$$

where, as above, $a = O(1)$. The last term, arising from the conformal region, through the second spectral function sum rule, is thus expected to be negative and of the same order as the first two terms. While this is a crude estimate of the S parameter, it seems that it could be much reduced relative to QCD-like theories.

Other attempts to estimate the S parameter for walking technicolor theories have been made in the past. In Ref. [14], based on an exotic method of analytic continuation whose accuracy seems to be difficult to estimate [15], it was claimed that the S parameter might be negative. In another approach [16], a single loop of fermions with a dynamically generated mass was used to estimate S directly, with no attempt to incorporate a low lying resonance spectrum. Since the S parameter is so dominated by the low momentum region, it seems important to incorporate the resonances in the estimate. They have played a central role in the results described here.

Finally, we remark that the low energy spectrum described here can be accommodated within the framework of an effective chiral Lagrangian. In the approximation we have employed, the spectrum consists of a set of Nambu-Goldstone bosons along with a multiplet of vector and axial particles. An appropriate, parity-invariant effective Lagrangian for this set of particles is the following [17]:

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[D_\mu M D^\mu M^\dagger] - \frac{1}{2}\mathrm{Tr}[F^L_{\mu\nu}F^{L\mu\nu} + F^R_{\mu\nu}F^{R\mu\nu}] + m_0^2\,\mathrm{Tr}[A^L_\mu A^{L\mu} + A^R_\mu A^{R\mu}] + h\,\mathrm{Tr}[A^L_\mu M A^{R\mu} M^\dagger]. \qquad (16)$$

The Nambu-Goldstone bosons are encoded in the $N_f \times N_f$ meson matrix M which transforms linearly under the chiral symmetry group $\mathrm{SU}_L(N_f)\otimes\mathrm{SU}_R(N_f)$, while $D_\mu M = \partial_\mu M - igA^L_\mu M + igMA^R_\mu$ is the chiral covariant derivative. The symmetry is realized nonlinearly through the constraint $MM^\dagger = M^\dagger M = F_\pi^2/2$. The third and fourth terms in the Lagrangian reduce the chiral symmetry from local to global, giving masses to the vector and axial vector mesons. Using the vacuum value $\langle M\rangle = \mathbf{1}F_\pi/\sqrt{2}$, we find [17] $M_A^2 - M_V^2 = \frac{1}{2}F_\pi^2[g^2 - h]$. Comparison with Eq. (13) shows that h parametrizes the contribution from the conformal region. It is important to point out that Eq. (16) is a nonlinear effective Lagrangian for the simplified spectrum we have considered, useful only in the broken phase.

In this Brief Report we have shown that the ordering pattern for vector–axial-vector hadronic states in SU(N) vector-like gauge theories close to a conformal transition need not be the same as predicted in QCD-like theories. To show this, we employed spectral function sum rules, the known asymptotic behavior determined by asymptotic freedom, and the fact that these theories contain an extended ''conformal region'' below the asymptotic regime.

A simple description of the conformal region was used to argue that it leads to a reduced and possibly even inverted vector-axial mass splitting. A possible consequence for technicolor theories was discussed.

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